An Analysis of (Linear) Exponentials Based on Extended Sequents

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Abstract

We apply the 2-sequents approach to the analysis of several calculi derived from linear logic. We present a uniform formal system for Linear Logic, Elementary Linear Logic and Light Linear Logic. In particular, the 2-sequent approach simplifies the syntax of Light and Elementary Linear Logic.

Keywords: Sequent Calculus. 2-sequents. Linear Logic. Polynomial cut-elimination. Elementary cut-elimination. Light Linear Logic. Elementary Linear Logic.

1 Introduction

The rationale for extending Gentzen's format for sequents is not unidimensional. It is often a blend of several issues that inspires the design of a particular system

[11, 2, 3, 16, 1]. The *2-sequent* approach [12, 13, 14, 15] is not an exception. Its original goal was notational: providing symmetric and local (i.e., context-free) rules for the minimal deontic logic KD. However, we discovered soon that 2-sequents could be used as a *uniform* tool for several logical systems. In particular, starting from a common core—the logical rules—we could shift from one system to another (e.g., from KD to S4) just changing the way in which syntactical objects are manipulated—say, the *structural rules*. In [15], we applied this scalar approach to the modal logics in the K-S4 range. Here, we shall apply the same methodology to Girard's Linear Logic (LL).

This study started in [14], where we gave a natural deduction style presentation

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for certain LL fragments. As a result, we discovered some unsuspected connections with proof-nets and—in view of the correspondence between λ-calculus and the multiplicative exponential fragment of LL (MELL)—with the so-called optimal or sharing implementation of lambda-terms (see [7, 9]). Here, we focus on full LL (i.e., with additives, second order quantifiers and constants) and on some subsystems with an intrinsic bound on the complexity of the representable functions. In particular we shall give a 2-sequent presentation of Light Linear Logic (LLL), see [6].

LLL is a "polynomial" logic: all the polynomial functions are representable by a (second order) LLL proof; moreover there is a suitable cut-elimination procedure that can be performed in polynomial time. The main drawback of LLL is its awkward syntax. The reduction bound of LLL is achieved by means of a tight control on the structure of the proof-nets associated to deductions, and on the way in which those nets grow along reduction. More precisely, the structural constraints on LLL proof-nets allow to reduce them by stages (i.e., pursuing an outermost-innermost strategy according to the nesting of exponential *boxes*), controlling at the same time the number of stages required to complete the task and the number of duplications performed at each stage. Unfortunately, mainly because of the presence of additives, the translation of that constraints on nets into logical rules leads to a system with an heavy ad hoc syntax, whose generalized sequents share very few structure with the standard ones of LL.

Because of the already mentioned tight relations between 2-sequents and proof-nets, it should not be particularly surprising that the structural constraints of LLL find instead a very simple formulation in the 2-sequent framework: the 2-sequent formulation of LLL (or 2LLL) is just the restriction of the 2-sequent formulation of LL (or 2LL) to the case in which only two levels are used, plus the new modality of LLL and the other structural restrictions on the auxiliary doors of boxes. In particular, the interaction between additives and exponentials requires no longer the introduction of a contrived syntax and two distinct formula separators in sequents (we will return on this point in section 5.4).

It is also remarkable that the same approach scales in a natural way to Elementary Linear Logic (ELL), an intermediate system between LLL and LL with (Kalmar) elementary cut-elimination.

Since the main focus of the paper is on the use of 2-sequents we will omit the dynamics of the proposed logical systems, that is indeed the relevant issue of LLL. At the same time, we omit for lack of space the definition and study of the indexed proof-nets corresponding to LLL and ELL.

2 2-sequents for linear logic

We refer to [4, 5] for notation and preliminaries on Linear Logic. We will write many times (even in formal sequents) the linear implication $A \multimap B$ instead of $A^{\perp} \parr B$.

Definition 2.1 (2-sequents) *A* 2-sequence *is an expression*

$$\begin{array}{c}\alpha_0\\ \alpha_1\\ \vdots\\ \alpha_n\end{array}$$

in which each α_i is an ordinary (possibly empty) sequence of linear formulas. The formulas of α_i are at level i. *A* 2-sequent *is an expression* $\vdash \Gamma$*, where* Γ *is a 2-sequence in which at least one of the α_i is not empty.*

Informally, levels express a form of modal (exponential) dependency.

Definition 2.2 (interpretation of 2-sequences) *The* interpretation $[\Gamma]^\sharp$ *of the 2-sequence* Γ *is defined as*

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}^\sharp = \parr\alpha_j \parr !(\parr\alpha_{j+1} \parr !(\ldots !(\parr\alpha_n)\ldots))$$

where j is the minimum level s.t. α_j is not empty; $\parr\alpha_k$ denotes the par of all the formulas in α_k (in particular, the term $\parr\alpha_k$ is missing when α_k is empty).

According to this interpretation, (modality related) structural rules naturally correspond to vertical rearrangement of formulas. Anyhow, since towers of sequences of formulas are not handy to write and manipulate, in the following we shall prefer an equivalent indexed representation. Namely, we will index any formula occurrence A in some α_i by its level i—say that A^i is the correspondent *indexed formula.* Any tower of α_i can then be merged into an ordinary linear *sequence of indexed formulas.* We will resort again to two-dimensional sequents for the relevant case of LLL (Section 5.1).

3 2LL: a 2-sequent calculus for Linear Logic

3.1 The calculus

The axioms and rules in Figure 1 define 2LL, whose provability relation will be denoted by $\vdash_{2LL}$. In the following sections, we shall prove its equivalence (with respect to provability) to the standard presentation of LL.

For the 2-sequence Γ, $\max(\Gamma) = \max\{i : A^i \in \Gamma\}$; moreover, we write $\Gamma^{=i}$ to mean that all the formulas in Γ are at level i. The $\forall$ rule has the usual proviso on the second order variable it binds: X must not occur free in any formula in Γ.

3.2 Correctness

In order to relate provability in 2LL to provability in standard LL ($\vdash_{LL}$), it is useful to isolate a class of formulas behaving as ?-modal formulas.

Definition 3.1 (essentially exponential formulas) *The class* Exp *is inductively defined as follows:*

1. $\perp \in$ Exp*;*
2. *for any formula A,* $?A \in$ Exp*;*
3. *if $A, B \in$ Exp, then $A \parr B, A \mathbin{\&} B \in$ Exp.*

The following is the main property of Exp (its proof is just an easy induction on the definition of Exp).

Identity/Negation

$$\vdash A^i, A^{\perp i} \qquad\qquad \dfrac{\vdash \Gamma, A^i \quad \vdash \Delta, A^{\perp i}}{\vdash \Gamma, \Delta}\,\mathsf{cut}_{\,i \geq \max(\Gamma),\max(\Delta)}$$

Structure

$$\dfrac{\vdash \Gamma}{\vdash \Gamma, ?A^i}\,W_{\,i \leq \max(\Gamma)} \qquad\qquad \dfrac{\vdash \Gamma, ?A^i, ?A^i}{\vdash \Gamma, ?A^i}\,C$$

Logic

$$\dfrac{\vdash \Gamma, A^i}{\vdash \Gamma, ?A^{i-j}}\,?_{\,i \geq j \geq 0} \qquad\qquad \dfrac{\vdash \Gamma, A^i}{\vdash \Gamma, !A^{i-1}}\,!_{\,i > \max(\Gamma)}$$

$$\dfrac{\vdash \Gamma, C^i \quad \vdash \Gamma, D^i}{\vdash \Gamma, C \mathbin{\&} D^i}\,\& \qquad \dfrac{\vdash \Gamma, C^i}{\vdash \Gamma, C \oplus D^i}\,\oplus_{\mathsf{L}\,i \geq \max(\Gamma)} \qquad \dfrac{\vdash \Gamma, D^i}{\vdash \Gamma, C \oplus D^i}\,\oplus_{\mathsf{R}\,i \geq \max(\Gamma)}$$

$$\dfrac{\vdash \Gamma, C^i \quad \vdash \Delta, D^i}{\vdash \Gamma, \Delta, C \otimes D^i}\,\otimes_{\,i \geq \max(\Gamma),\max(\Delta)} \qquad\qquad \dfrac{\vdash \Gamma, C^i, D^i}{\vdash \Gamma, C \parr D^i}\,\parr$$

$$\dfrac{\vdash \Gamma, A^i}{\vdash \Gamma, \forall X.A^i}\,\forall_{\,i \geq \max(\Gamma)} \qquad\qquad \dfrac{\vdash \Gamma, A^i[B/X]}{\vdash \Gamma, \exists X.A^i}\,\exists_{\,i \geq \max(\Gamma)}$$

$$\dfrac{\vdash \Gamma}{\vdash \Gamma, \perp^i}\,\perp_{\,i \leq \max(\Gamma)} \qquad\qquad \vdash 1^i \qquad\qquad \vdash \Gamma^{=i}, \top^i$$

FIG. 1. 2LL: 2-sequent presentation of LL

Lemma 3.2 *For any* $A \in \mathsf{Exp}$*,* $\vdash_{\mathsf{LL}} A, !A^{\perp}$.

Therefore, weakening and contraction are admissible on formulas in Exp, and, moreover, the promotion rule with Exp-contexts (i.e., contexts in which all the formulas are in Exp) is derivable.

Lemma 3.3 *Let* $\Gamma = A_1, \ldots, A_n$ *be a (standard) sequent such that any* $A_i \in \mathsf{Exp}$*. Then*

$$\vdash_{\mathsf{LL}} \Gamma, A \quad \Rightarrow \quad \vdash_{\mathsf{LL}} \Gamma, !A$$

We may use 2LL to show that many formulas are in Exp.

Lemma 3.4 *Let* $\vdash_{\mathsf{2LL}} \Gamma$*. For any* $A^i \in \Gamma$ *such that* $i < \max(\Gamma)$*,* $A \in \mathsf{Exp}$*.*

PROOF. By induction on the derivation, exploiting the side-conditions. ∎

Indeed, we could constrain the rules $\parr$, $\&$ and $\perp$ to be applied only at maximal levels (like $\forall$, $\exists$, $\otimes$, $\oplus$ and cut) without losing in provability. In this more restrained system, all the non-maximal formulas of a provable 2-sequent are of the shape $?A$.

Correctness of 2LL with respect to LL is now an easy corollary, since any 2LL rule may be "flattened" to its corresponding LL rule. Let us define the following new interpretation of 2-sequences, simpler than the one presented in Definition 2.2:

$$\left[\begin{array}{c}\alpha_0\\ \alpha_1\\ \vdots\\ \alpha_n\end{array}\right]^\flat = \alpha_0,\ldots,\alpha_n$$

Theorem 3.5 (correctness)

$$\vdash_{\mathsf{2LL}} \Gamma \quad\Rightarrow\quad \vdash_{\mathsf{LL}} [\Gamma]^\flat$$

Proof. Any 2LL rule but promotion (!) becomes the corresponding LL rule via the $\flat$-translation. Promotion is instead handled by Lemma 3.3, since by the side condition and Lemma 3.4, $[\Gamma]^\flat$ is an Exp-context. ∎

The reader might wonder why we gave Definition 2.2 if we rather used the $\flat$-translation when proving correctness. The reason is that the $\flat$-translation, though correct, hides the level "semantics." Moreover, expressing LL in terms of proof-nets, the levels have a very intuitive notion, they are the box nesting depth of each formula. It is clear, however, that the two translations are related, as shown by the following proposition.

Proposition 3.6 *Let Γ be a 2-sequence such that, for any $A^i \in \Gamma$ with $i < \max(\Gamma)$, $A \in \mathsf{Exp}$. Then*

$$\vdash_{\mathsf{LL}} [\Gamma]^\sharp \multimap \parr [\Gamma]^\flat \qquad\qquad \textit{and} \qquad\qquad \vdash_{\mathsf{LL}} \parr [\Gamma]^\flat \multimap [\Gamma]^\sharp$$

Proof. The implication $[\Gamma]^\sharp \multimap \parr [\Gamma]^\flat$ follows from the comonad law $\vdash_{\mathsf{LL}} !A \multimap A$. For the other direction, we use repeatedly Lemmas 3.4 and 3.3. ∎

Observe that, in view of Lemma 3.4, the hypotheses of the previous proposition hold for any *provable* 2-sequence.

3.3 Completeness

Lemma 3.7 *The following rule is admissible in* 2LL*:*

$$\frac{\vdash ?\Gamma^{=0}, A^0}{\vdash ?\Gamma^{=0}, !A^0}$$

Proof. Observe first that, whenever $\vdash_{\mathsf{2LL}} ?\Gamma^{=0}, A^0$, we also have $\vdash_{\mathsf{2LL}} ?\Gamma^{=1}, A^1$ (simply add one to each level in the proof). Now, with several ? rules we obtain $\vdash ??\Gamma^{=0}, A^1$, and then $\vdash ??\Gamma^{=0}, !A^0$. Finally, we cut this sequent against the provable sequent $\vdash !!\Gamma^{\perp=0}, ?\Gamma^{=0}$. ∎

Theorem 3.8 (completeness)

$$\vdash_{\mathsf{LL}} \alpha \quad\Rightarrow\quad \vdash_{\mathsf{2LL}} \alpha^{=0}$$

PROOF. By induction on the length of the proof of $\vdash_{\mathsf{LL}} \alpha$ and by cases on the last rule. Most cases are trivial. For instance, the ? rule is handled directly taking $j = 0$; for the ! rule, we apply Lemma 3.7; and so on. ∎

A careful inspection of the proof above shows that all the non-modal rules are applied with the principal formulas at maximal levels. Therefore, the addition of this restriction would not have any impact on completeness.

Corollary 3.9 (equivalence)

$$\vdash_{\mathsf{LL}} \alpha \quad \Leftrightarrow \quad \vdash_{\mathsf{2LL}} \alpha^{=0}$$

4 Taming the complexity of LL

Following the ideas in [14], several subsystems of LL can be obtained by constraining the range of variation of the j parameter in the ? rule. In fact:

1. setting $i \geq j > 0$, we avoid the principle $!A \multimap A$;
2. setting $j \in \{0, 1\}$, we avoid the principle $!A \multimap !!A$;
3. with $j = 1$, we get rid of both the previous principles.

Remark 4.1 *Let* 2ELL *be the system in which the* ? *rule is applied always with* $j = 1$. *The fact that* 2ELL *does not prove* $!A \multimap !!A$ *and* $!A \multimap A$ *is established by a simple translation of linear formulas into classical modal ones: linear propositional connectives are replaced with classical conjunction and disjunction,* ! *with necessity and* ? *with possibility (see also section 4.1). The system* 2ELL *is translated in this way into a sub-calculus of the 2-sequent calculus for* KD *(see [12]). Obviously, if* 2ELL *proved* $!A \multimap !!A$ *or* $!A \multimap A$*, the 2-sequent calculus for* KD *would do the same for* $\Box A \rightarrow \Box\Box A$ *and* $\Box A \rightarrow A$*, which is instead impossible.*

The interest of these subsystems stems from the fact that they avoid the rules that are the main culprits for the super-exponential cost of cut-elimination for LL. In fact, questing for a logical system with an intrinsic polynomial complexity (i.e., with a polynomial cut-elimination ensuring at the same time that all polynomial functions are representable) Girard proposed in [6] the Light Linear Logic (LLL) system, dropping, among others, the laws mentioned above (see the appendix for the rules of LLL). The purpose of this section is to reconstruct LLL in a 2-sequent notation.

In the same paper, Girard briefly sketched ELL (Elementary Linear Logic), a super-system of LLL with a (Kalmar) elementary cut-elimination (see the appendix for the rules of ELL). Also ELL is easily formulated in our notation: it is simply 2ELL, once Girard's syntax (especially what he calls "blocks") is expressed in the 2-sequent language. The equivalence between ELL and 2ELL will be stated after the section devoted to LLL, since ELL shares most of the syntax with these systems. Before getting through LLL, however, let us pause for a moment, discussing why (presumably) ELL was presented with a complicated syntax. A complication that disappears in 2ELL.

4.1 ELL *and the deontic logic* KD

From the modal point of view, ELL is akin to KD (see Remark 4.1). Sequent rules for KD are well known; the standard ones are the following, where both modalities are introduced *at once*:

$$\frac{\vdash \Gamma, A}{\vdash \Diamond\Gamma, \Box A} \qquad \frac{\vdash \Gamma}{\vdash \Diamond\Gamma}$$

It is this "one-shot" formulation of modalities that eliminates the laws $\Box A \to \Box\Box A$ and $\Box A \to A$.

In linear logic, the same target cannot be reached so trivially because of the interaction between the exponential and the propositional connectives (actually, mainly because of the additives). It is to avoid the modification of the usual formulation of linear logic propositional rules that Girard introduced two distinct formula separators in ELL and LLL sequents: the ";" with the usual multiplicative intended meaning (i.e., it can be internalized by the $⅋$ connective) and the "," with an additive intended meaning (i.e., it can be internalized by the $\oplus$ connective). As we shall see in more detail in section 5.4, the use of 2-sequents allows to avoid all these kind of problems and to formulate ELL and LLL in a "more traditional" way.

5 Light Linear Logic

ELL is still too powerful. Many of its rules must be restricted to capture polynomial time. From an axiomatic point of view, many laws valid in ELL have to be abandoned; among these, the generalization rule:

$$\frac{\vdash A}{\vdash !A}\ \mathsf{gen}$$

the law $!(A \multimap B) \multimap (!A \multimap !B)$ and the law $!A \otimes !B \multimap !(A \otimes B)$. Moreover, in order to restore some of the lost power in a harnessed way, a new auto-dual modality § must be added to the system.

In the 2-sequent framework, many constraints have to be added to 2ELL. First, the rule for the ! must be formulated in order to avoid generalization as a special case:

$$\frac{\vdash \Gamma, A^i}{\vdash \Gamma, !A^{i-1}}\ !_{i > \max(\Gamma), \Gamma \neq \varnothing}$$

Second, we must avoid the two laws mentioned above, amounting to a drastic restriction of cut and $\otimes$. Namely, all the premises (and conclusions) must be at the same level. For the same reason, also the ? rule has to be strongly harnessed, asking the context to consist of exactly one formula:

$$\frac{\vdash B^i, A^i}{\vdash B^i, ?A^{i-1}}\ ?$$

Third, weakening must be restricted:

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A^i}\ W_{i \leq \max(\Gamma), \Gamma^{\leq i} \neq \varnothing}$$

Finally, the new auto-dual modality ($(\S A)^\perp = \S A^\perp$) is regulated by the single rule:

$$\frac{\vdash \Gamma^{=i+1}, \Delta^{=i+1}}{\vdash ?\Gamma^{=i}, \S\Delta^{=i}}\ ?\S$$

which makes clear that § stays midway between ! and ?. In fact, $\vdash_{\mathsf{LLL}} !A \multimap \S A$ and $\vdash_{\mathsf{LLL}} \S A \multimap ?A$.

All these restrictions make (essentially) useless the abundance of levels of the calculus. In fact, in the following section we present an equivalent (and we believe more elegant) calculus with only two levels, to which we reserve the official name 2LLL. The equivalence of 2LLL to the sketched formulation with many levels is left to the interested reader and may be proved along the lines of Theorem 5.3.

5.1 2LLL

The formulation of 2LLL resorting to the two-dimensional notation for 2-sequents is given in Figure 2. All the sequences have at most two levels; recall that α, β range over ordinary sequences of formulas.

We prove next that 2LLL is equivalent to the standard formulation of Girard. In the following, $A, B, A_i, \ldots$ range over formulas; $\mathbf{A}, \mathbf{B}, \mathbf{A}_i \ldots$ range over blocks. Moreover, the block $A_1, \ldots, A_n$ stands for (or to put it as in [6] "is hypocrisy for") $A_1 \oplus \ldots \oplus A_n$; while, if $\mathbf{A}_1, \ldots, \mathbf{A}_n$ stands for the formulas $A_1, \ldots, A_n$, the sequence $\mathbf{A}_1; \ldots; \mathbf{A}_n$ stands for $A_1 \parr \ldots \parr A_n$.

5.2 Soundness

We slightly modify the interpretation of Definition 2.2 to take into account the block notion.

Definition 5.1 (interpretation of LLL 2-sequences)

1. $\overline{A_1, \ldots, A_n} = A_1; \ldots; A_n$;
2. $\parr(A_1, \ldots, A_n) = A_1 \parr \ldots \parr A_n$;
3. $[\Gamma]^{\circledcirc} = \begin{cases} [\alpha]^{\circledcirc} = \overline{\alpha} & \text{when} \quad \Gamma = \alpha \\ \begin{bmatrix} \alpha \\ \beta \end{bmatrix}^{\circledcirc} = \overline{\alpha}; ! \parr \beta & \text{when} \quad \Gamma = \begin{matrix} \alpha \\ \beta \end{matrix} \end{cases}$

In the following, we will use several times and without explicit reference the following facts:

1. $\vdash_{\mathsf{LLL}} A \parr B$ and $\vdash_{\mathsf{LLL}} B \multimap C$ implies $\vdash_{\mathsf{LLL}} A \parr C$;
2. $\vdash_{\mathsf{LLL}} !B$ whenever $\vdash_{\mathsf{LLL}} A \multimap B$ and $\vdash_{\mathsf{LLL}} !A$.

In particular, we recall that the second fact replaces $!(A \multimap B) \multimap (!A \multimap !B)$, which *is false* in LLL.

Lemma 5.2 $\vdash_{\mathsf{LLL}} !(B \parr C) \otimes !(B \parr D) \multimap !(B \parr (C \,\&\, D))$.

Identity/Negation

$$\vdash\ A, A^\perp \qquad\qquad \dfrac{\vdash\ \alpha, A \qquad \vdash\ \beta, A^\perp}{\vdash\ \alpha, \alpha}\,\mathsf{cut}$$

Structure

$$\dfrac{\vdash\ \begin{array}{c}\alpha\\ \beta\end{array}}{\vdash\ \begin{array}{c}\alpha\\ \beta, ?A\end{array}}\,W1_{\ \beta\neq\varnothing} \qquad \dfrac{\vdash\ \begin{array}{c}\alpha\\ \beta\end{array}}{\vdash\ \begin{array}{c}\alpha, ?A\\ \beta\end{array}}\,W2_{\ \alpha\neq\varnothing} \qquad \dfrac{\vdash\ \begin{array}{c}\alpha\\ \beta, ?A, ?A\end{array}}{\vdash\ \begin{array}{c}\alpha\\ \beta, ?A\end{array}}\,C1 \qquad \dfrac{\vdash\ \begin{array}{c}\alpha, ?A, ?A\\ \beta\end{array}}{\vdash\ \begin{array}{c}\alpha, ?A\\ \beta\end{array}}\,C2$$

Logic

$$\dfrac{\vdash\ \alpha, \beta}{\vdash\ ?\alpha, \S\beta}\,?\S \qquad \dfrac{\vdash\ A, B}{\vdash\ \begin{array}{c}?A\\ B\end{array}}\,? \qquad \dfrac{\vdash\ \begin{array}{c}\alpha\\ A\end{array}}{\vdash\ \alpha, !A}\,!_{\ \alpha\neq\varnothing}$$

$$\dfrac{\vdash\ \begin{array}{c}\alpha\\ \beta, C\end{array} \qquad \vdash\ \begin{array}{c}\alpha\\ \beta, D\end{array}}{\vdash\ \begin{array}{c}\alpha\\ \beta, C \,\&\, D\end{array}}\,\& \qquad \dfrac{\vdash\ \begin{array}{c}\alpha\\ \beta, C\end{array}}{\vdash\ \begin{array}{c}\alpha\\ \beta, C \oplus D\end{array}}\,\oplus_{\mathsf{L}} \qquad \dfrac{\vdash\ \begin{array}{c}\alpha\\ \beta, D\end{array}}{\vdash\ \begin{array}{c}\alpha\\ \beta, C \oplus D\end{array}}\,\oplus_{\mathsf{R}}$$

$$\dfrac{\vdash\ \alpha, C \qquad \vdash\ \beta, D}{\vdash\ \alpha, \beta, C \otimes D}\,\otimes \qquad \dfrac{\vdash\ \begin{array}{c}\alpha\\ \beta, C, D\end{array}}{\vdash\ \begin{array}{c}\alpha\\ \beta, C \parr D\end{array}}\,\parr_1 \qquad \dfrac{\vdash\ \begin{array}{c}\alpha, C, D\\ \beta\end{array}}{\vdash\ \begin{array}{c}\alpha, C \parr D\\ \beta\end{array}}\,\parr_2$$

$$\dfrac{\vdash\ \alpha, A}{\vdash\ \alpha, \forall X.A}\,\forall \qquad\qquad \dfrac{\vdash\ \alpha, A}{\vdash\ \alpha, \exists X.A}\,\exists$$

$$\dfrac{\vdash\ \alpha}{\vdash\ \alpha, \bot}\,\bot \qquad\qquad \vdash\ 1 \qquad\qquad \vdash\ \alpha, \top$$

FIG. 2. 2LLL: 2-sequent presentation of LLL.

PROOF.

$$\begin{array}{c}
\dfrac{\dfrac{\dfrac{\vdash B^\perp;\ B \quad \vdash C^\perp;\ C}{\vdash B^\perp \otimes C^\perp;\ B;\ C}}{\vdash B^\perp \otimes C^\perp, B^\perp \otimes D^\perp;\ B;\ D} \qquad \dfrac{\dfrac{\vdash B^\perp;\ B \quad \vdash D^\perp;\ D}{\vdash B^\perp \otimes D^\perp;\ B;\ D}}{\vdash B^\perp \otimes C^\perp, B^\perp \otimes D^\perp;\ B;\ D}}{\vdash B^\perp \otimes C^\perp, B^\perp \otimes D^\perp;\ B;\ C \,\&\, D} \\ \hline
\vdash B^\perp \otimes C^\perp, B^\perp \otimes D^\perp;\ B \parr C \,\&\, D \\ \hline
\vdash [B^\perp \otimes C^\perp];\ [B^\perp \otimes D^\perp];\ !(B \parr (C \,\&\, D)) \\ \hline
\vdash ?(B^\perp \otimes C^\perp);\ [B^\perp \otimes D^\perp];\ !(B \parr (C \,\&\, D)) \\ \hline
\vdash ?(B^\perp \otimes C^\perp);\ ?(B^\perp \otimes D^\perp);\ !(B \parr (C \,\&\, D)) \\ \hline
\vdash ?(B^\perp \otimes C^\perp) \parr\, ?(B^\perp \otimes D^\perp);\ !(B \parr (C \,\&\, D)) \\ \hline
\vdash !(B \parr C) \otimes !(B \parr D) \multimap !(B \parr (C \,\&\, D))
\end{array}$$

■

Theorem 5.3 (soundness of 2LLL)

$$\vdash_{\mathsf{2LLL}} \Gamma \quad \Rightarrow \quad \vdash_{\mathsf{LLL}} [\Gamma]^{\circledcirc}$$

Proof. By induction on the proof of $\vdash_{\mathsf{2LLL}} \Gamma$ and by cases on the last rule. The case of axiom, cut, $\otimes$, $\forall$, $\exists$, $\bot$, 1, and $\top$ are obvious, since they are interpreted into the corresponding LLL rules. The cases of ! and $\parr$ are also trivial, since their premises and conclusions have the same interpretation. For the other rules we have instead:

Rules $\oplus$: Easy, since $\vdash_{\mathsf{2LLL}} C \multimap C \oplus D$ and $\vdash_{\mathsf{2LLL}} C \multimap D \oplus C$.

Rule §: We apply the neutral rule first and then the why-not rule many times.

Rule ?: We apply the of-course rule first and then the why-not rule once.

Rule &: If α is empty, we use the with rule of LLL. Otherwise, the premises of the rule are interpreted as $\overline{\alpha}$; $!(\parr\overline{\beta} \parr C)$ and $\overline{\alpha}$; $!(\parr\overline{\beta} \parr D)$. Thus, by induction hypothesis, $\vdash_{\mathsf{LL}}$ $\overline{\alpha}$; $\overline{\alpha}$; $!(\parr\overline{\beta} \parr C) \otimes !(\parr\overline{\beta} \parr D)$. By a cut, using Lemma 5.2, we obtain $\vdash_{\mathsf{LL}}$ $\overline{\alpha}$; $\overline{\alpha}$; $!(\parr\overline{\beta} \parr (C \,\&\, D))$. We observe now that, for $\overline{\alpha} = A_1; \ldots; A_n$, any A_i is (in the class Exp and in particular is) of the form $?B_1 \parr \ldots \parr ?B_{k_i}$. Therefore, using $\vdash_{\mathsf{LL}} ?B \parr ?B \multimap ?B$, by several successive cuts we obtain the thesis.

Rule W1: By $\vdash_{\mathsf{LL}} B \multimap B \parr ?A$.

Rule W2: We use first mult-W, and then the why-not rule of LLL.

Rules C1 and C2: We use again $\vdash_{\mathsf{LL}} ?B \parr ?B \multimap ?B$.

■

5.3 Completeness

Definition 5.4 (Translation of LLL sequents) *Let $A, A_1, \ldots A_n$ be formulas and $\mathbf{A}, \mathbf{A}_1, \ldots \mathbf{A}_m$ be blocks:*

1. $(A)^{\circledast} = A$ *if A is a formula;*
2. $([A])^{\circledast} = ?A$*;*
3. $(A1, \ldots, A_n)^{\circledast} = A1 \oplus \ldots \oplus A_n$*;*
4. $(\mathbf{A}_1; \ldots; \mathbf{A}_n)^{\circledast} = (\mathbf{A}_1)^{\circledast}, \ldots, (\mathbf{A}_n)^{\circledast}$.

Lemma 5.5

$$\vdash_{\mathsf{2LLL}} ?B_1, \ldots, ?B_n, !(B_1^{\perp} \,\&\, \cdots \,\&\, B_n^{\perp})$$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash B_1, B_1^{\perp}}{\vdash \begin{array}{c}?B_1\\ B_1^{\perp}\end{array}}\,?}{\vdots}\,W2}{\vdash \begin{array}{c}?B_1, \ldots, ?B_n\\ B_1^{\perp}\end{array}}\,W2 \qquad \dfrac{\begin{array}{c}\vdash B_2, B_2^{\perp}\\ \vdots\\ \vdash \begin{array}{c}?B_1, \ldots, ?B_n\\ B_2^{\perp}\end{array}\end{array} \qquad \begin{array}{c}\vdash B_3, B_3^{\perp} \;\cdots \vdash B_n, B_n^{\perp}\\ \vdots\\ \vdash \begin{array}{c}?B_1, \ldots, ?B_n\\ B_3^{\perp} \,\&\, \cdots \,\&\, B_n^{\perp}\end{array}\end{array}}{\vdash \begin{array}{c}?B_1, \ldots, ?B_n\\ B_2^{\perp} \,\&\, B_3^{\perp} \,\&\, \cdots \,\&\, B_n^{\perp}\end{array}}\,\&}{\dfrac{\vdash \begin{array}{c}?B_1, \ldots, ?B_n\\ B_1^{\perp} \,\&\, B_2^{\perp} \,\&\, \cdots \,\&\, B_n^{\perp}\end{array}}{\vdash ?B_1, \ldots, ?B_n, !(B_1^{\perp} \,\&\, B_2^{\perp} \,\&\, \cdots \,\&\, B_n^{\perp})}\,!}\,\&$$

■

Theorem 5.6 (completeness of 2LLL)

$$\vdash_{\mathsf{LLL}} \mathbf{A}_1; \ldots; \mathbf{A}_m \quad \Rightarrow \quad \vdash_{\mathsf{2LLL}} (\mathbf{A}_1)^{\circledast}, \ldots, (\mathbf{A}_m)^{\circledast}$$

PROOF. By induction on the proof of $\vdash_{\mathsf{LLL}} \mathbf{A}_1; \ldots; \mathbf{A}_m$ and by cases on the last rule. All the non-modal rules of LLL (plus why-not) are handled trivially. The only interesting cases are those of of-course and neutral.
In the original formulation of LLL the of-course rule is:

$$\dfrac{\vdash B_1, \ldots, B_n; A}{\vdash [B_1], \ldots, [B_n]; !A}\ \textsf{of-course}$$

By definition, $(B_1, \ldots, B_n; A)^{\circledast} = B_\oplus, A$, where $B_\oplus = B_1 \oplus \cdots \oplus B_n$. By induction hypothesis, $\vdash_{\mathsf{2LLL}} B_\oplus, A$. Thus, since by definition we have that $([B_1], \ldots, [B_n]; !A)^{\circledast} = ?B_1, \ldots, ?B_n, !A$, the required 2LLL proof is:

$$\dfrac{\dfrac{\text{Lemma 5.5}}{\vdash\ ?B_1, \ldots, ?B_n, !B_\oplus^\perp} \qquad \dfrac{\dfrac{\vdash\ B_\oplus, A}{\vdash\ \begin{matrix}?B_\oplus\\ A\end{matrix}}\ ?}{\vdash\ ?B_\oplus, !A}\ !}{\vdash\ ?B_1, \ldots, ?B_n, !A}\ \textsf{cut}$$

In the original formulation of LLL the neutral rule is:

$$\dfrac{\vdash C^1_1, \ldots, C^1_{p_1}; \ldots; C^n_1, \ldots, C^n_{p_n}; A_1; \ldots; A_m}{\vdash [C^1_1]; \ldots; [C^1_{p_1}]; \ldots; [C^n_1]; \ldots; [C^n_{p_n}]; \S A_1; \ldots; \S A_m}\ \textsf{neutral}$$

Let $C^k_\oplus = C^k_1 \oplus \cdots \oplus C^k_{p_k}$ and $\widetilde{C}^k_\& = (C^k_1)^\perp \,\&\, \cdots \,\&\, (C^k_{p_k})^\perp = (C^k_\oplus)^\perp$. The premise becomes $(C^1_1, \ldots, C^1_{p_1}; \ldots; C^n_1, \ldots, C^n_{p_n}; A_1; \ldots; A_m)^{\circledast} = C^1_\oplus, \ldots, C^n_\oplus, A_1, \ldots, A_m$. By induction hypothesis, $\vdash_{\mathsf{2LLL}} C^1_\oplus, \ldots, C^n_\oplus, A_1, \ldots, A_m$. Therefore, since the conclusion becomes $([C^1_1]; \ldots; [C^n_{p_n}]; \S A_1; \ldots; \S A_m)^{\circledast} = ?C^1_1, \ldots, ?C^n_{p_n}, \S A_1, \ldots, \S A_m$, the required 2LLL proof is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash\ C^1_\oplus, \ldots, C^n_\oplus, A_1 \ldots, A_m}{\vdash\ ?C^1_\oplus, \ldots, ?C^n_\oplus, \S A_1, \ldots, \S A_m}\ \S \qquad \dfrac{\text{Lemma 5.5}}{\vdash\ ?C^1_1, \ldots, ?C^1_{p_1}, !\widetilde{C}^1_\&}}{\vdash\ ?C^1_1, \ldots, ?C^1_{p_1}, ?C^2_\oplus \ldots, ?C^n_\oplus, \S A_1, \ldots, \S A_m}\ \textsf{cut} \qquad \vdots}{\vdots}\ \textsf{cut} \qquad \vdots}{\vdash\ ?C^1_1, \ldots, ?C^n_{p_n}, \S A_1, \ldots, \S A_m}\ \textsf{cut}$$

■

Corollary 5.7 (equivalence)

$$\vdash_{\mathsf{2LLL}} \mathbf{A}_1; \ldots; \mathbf{A}_m \quad \Longleftrightarrow \quad \vdash_{\mathsf{2LLL}} (\mathbf{A}_1; \ldots; \mathbf{A}_m)^{\circledast}.$$

5.4 Intermezzo: Proofs in 2LLL

Performing a proof in 2LLL is "more natural" than constructing a proof of the same conclusion in LLL. The argument is best exemplified by comparing the formula

$$?A \parr (?B \parr !(A^\perp \mathbin{\&} B^\perp)),$$

which is provable in LLL. Its LL proof is:

$$
\begin{prooftree}
\AxiomC{$\vdash A, A^\perp$}
\RightLabel{$?$}
\UnaryInfC{$\vdash ?A, A^\perp$}
\RightLabel{W}
\UnaryInfC{$\vdash ?A, ?B, A^\perp$}
\AxiomC{$\vdash B, B^\perp$}
\RightLabel{$?$}
\UnaryInfC{$\vdash ?B, B^\perp$}
\RightLabel{W}
\UnaryInfC{$\vdash ?A, ?B, B^\perp$}
\RightLabel{$\&$}
\BinaryInfC{$\vdash ?A, ?B, A^\perp \mathbin{\&} B^\perp$}
\RightLabel{$!$}
\UnaryInfC{$\vdash ?A, ?B, !(A^\perp \mathbin{\&} B^\perp)$}
\RightLabel{$\parr$}
\UnaryInfC{$\vdash ?A, ?B \parr !(A^\perp \mathbin{\&} B^\perp)$}
\RightLabel{$\parr$}
\UnaryInfC{$\vdash ?A \parr (?B \parr !(A^\perp \mathbin{\&} B^\perp))$}
\end{prooftree}
$$

We see that it is *essential* to have separate rules for the two exponential connectives (i.e., we cannot merge their rules as in the deontic logic KD, see Section 4.1), for the introduction of ! is subordinated to a ? context built by two distinct ? rules (and, moreover, these two ? rules are on two distinct branches).

It is to take into account the need for two separate rules for ! and ?, that Girard was forced to adopt a rather contrived syntax for LLL. In particular, he chose to have a "liberal" rule for the introduction of !

$$\frac{\vdash B_1, \ldots, B_n;\ A}{\vdash [B_1]; \ldots; [B_n];\ !A}\ \text{of-course}$$

where no side-condition on the context of the premise is issued. But the context is modified *after* the introduction. Commas are replaced by semicolons and formulas by discharged formulas. The liberal rule for ! introduction is not constrained by the syntactical structure of the context. On the contrary, *it changes the context*. Therefore, the proof of our formula in Girard's LLL is the following:

$$
\begin{prooftree}
\AxiomC{$\vdash A; A^\perp$}
\RightLabel{add-W}
\UnaryInfC{$\vdash A, B; A^\perp$}
\AxiomC{$\vdash B; B^\perp$}
\RightLabel{add-W}
\UnaryInfC{$\vdash A, B; B^\perp$}
\RightLabel{with}
\BinaryInfC{$\vdash A, B; A^\perp \mathbin{\&} B^\perp$}
\RightLabel{of-course}
\UnaryInfC{$\vdash [A]; [B]; !(A^\perp \mathbin{\&} B^\perp)$}
\RightLabel{why-not}
\UnaryInfC{$\vdash [A]; ?B; !(A^\perp \mathbin{\&} B^\perp)$}
\RightLabel{why-not}
\UnaryInfC{$\vdash ?A; ?B; !(A^\perp \mathbin{\&} B^\perp)$}
\RightLabel{par}
\UnaryInfC{$\vdash ?A; ?B \parr !(A^\perp \mathbin{\&} B^\perp)$}
\RightLabel{par}
\UnaryInfC{$\vdash ?A \parr (?B \parr !(A^\perp \mathbin{\&} B^\perp))$}
\end{prooftree}
$$

where:

1. There are two additive weakenings (add-W).

2. Before the introduction of ! the intended meaning of the context is $A \oplus B$; after, it is $?A \parr ?B$. In a sense, there is an implicit use of the fact $A \oplus B \multimap ?A \parr ?B$.
3. Only after the introduction of ! we may change the discharged formulas $[A]$ and $[B]$ into why-not formulas.

In our 2ELL, exponentials are handled as first/second order quantifiers. There are two separate rules for ! and ?; and, moreover, the introduction of an exponential connective *does not change* the context. Finally, the introduction of ! is not free: its constraint is similar to the one of a universal quantifier (for a detailed discussion of this analogy see [8]).

By these properties, the skeleton of the proof of $?A \parr (?B \parr !(A^{\perp} \& B^{\perp}))$ in 2LLL is the same as that of LL:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash A, A^{\perp}}{\vdash \begin{array}{c}?A\\ A^{\perp}\end{array}}\,?}{\vdash \begin{array}{c}?A, ?B\\ A^{\perp}\end{array}}\,W2 \qquad \dfrac{\dfrac{\vdash B, B^{\perp}}{\vdash \begin{array}{c}?B\\ B^{\perp}\end{array}}\,?}{\vdash \begin{array}{c}?A, ?B\\ B^{\perp}\end{array}}\,W2}{\vdash \begin{array}{c}?A, ?B\\ A^{\perp} \& B^{\perp}\end{array}}\,\&_1}{\vdash ?A, ?B, !(A^{\perp} \& B^{\perp})}\,!}{\vdash ?A, ?B \parr !(A^{\perp} \& B^{\perp})}\,\parr_1}{\vdash ?A \parr (?B \parr !(A^{\perp} \& B^{\perp}))}\,\parr_1
$$

Next section will prove this in a more general setting (see Theorem 6.3).

6 Elementary Linear Logic

In Remark 4.1, we defined the system 2ELL as the restriction of 2LL obtained forcing $j = 1$ in the ? rule. As in the case of 2LLL, this amounts to a system with only two levels: the two-dimensional formulation of 2ELL is obtained from the one of 2LLL by removing the constraints on the rules for ? and !. In more detail, the rules of 2ELL are the ones in Figure 2 with the exception of the ?§ rule—§ does not exist in ELL—and of the ? and ! rules, which are replaced by:

$$
\dfrac{\vdash \begin{array}{c}\alpha\\ A, \beta\end{array}}{\vdash \begin{array}{c}\alpha, ?A\\ \beta\end{array}}\,? \qquad\qquad \dfrac{\vdash \begin{array}{c}\alpha\\ A\end{array}}{\vdash \alpha, !A}\,!
$$

The translation of 2ELL sequents into ELL sequents is the same as the one given for 2LLL. Moreover, completeness is proved in the same way.

Theorem 6.1 (completeness of 2ELL)

$$
\vdash_{\mathsf{ELL}} \mathbf{A}_1; \ldots; \mathbf{A}_m \quad \Rightarrow \quad \vdash_{\mathsf{2ELL}} (\mathbf{A}_1)^{\circledast}, \ldots, (\mathbf{A}_m)^{\circledast}
$$

6.1 Relating ELL and LL proofs

ELL is a subsystem of LL, in the sense that $\vdash_{\mathsf{LL}} A$ whenever $\vdash_{\mathsf{ELL}} A$. However, because of the *ad hoc* syntax of ELL, there is no direct correspondence between ELL proofs of a given formula and LL proofs of the same formula.

In the case of 2ELL (and even of the subsystem of 2LLL without the § modality) there is instead a straightforward way in which any 2ELL can be seen as a LL proof (in which the sequents have been split into an upper and a lower part).

Formally, consider again the "flattening" function $[\cdot]^\flat$:

$$[\alpha]^\flat = \alpha \qquad\qquad \begin{bmatrix} \alpha \\ \beta \end{bmatrix}^\flat = \alpha, \beta$$

For the sake of clarity, we take a restricted version of 2ELL where the rule $\wp_2$ is omitted. In the rest of this section, we will use 2ELL to the denote such a system (summarizing, in this section 2ELL is 2LLL minus the ?§ and $\wp_2$ rules, and without restrictions on the ? and ! rules). It is straigthforward to chech that even with this restriction the system remains complete for ELL (just observe that rule $\wp_2$ is never used in the proof of the completeness theorem 6.1).

As 2ELL is trivially a subsystem of 2LL, the following proposition is still valid:

Theorem 6.2

$$\vdash_{\mathsf{2ELL}} \Gamma \quad \Rightarrow \quad \vdash_{\mathsf{LL}} [\Gamma]^\flat$$

Extend now the $\flat$-translation to proofs: for a 2ELL proof Π, let $[\Pi]^\flat$ the tree obtained by replacing each 2-sequent $\vdash \Gamma$ in Π with the classical sequent $\vdash [\Gamma]^\flat$. This translation clearly maps 2ELL proofs into LL proofs.

Theorem 6.3 *For any* 2ELL *proof* Π*, the tree* $[\Pi]^\flat$ *is an* LL *proof.*

PROOF. It is almost immediate to note that, after the application of the $\flat$-translation, all the rules of 2ELL become rules of LL. The only case that deserves some consideration is the ! rule:

$$\frac{\vdash \begin{matrix} \alpha \\ A \end{matrix}}{\vdash \ \alpha, !A}\,!$$

Since Π does not contain $\wp_2$ rules, no formula in α is of the shape $A_1 \wp A_2$. By inspection of the rules we see that α is a sequence of ?-formulas. Hence,

$$\frac{\vdash \ \alpha, A}{\vdash \ \alpha, !A}$$

is a valid application of the promotion rule of LL. ∎

Summing up, when proving in 2ELL/2LLL, we use classical principles only, without annoying ourselves with complicated syntactical constraints. Moreover, there is a trivial characterization of the LL proofs corresponding to proofs of ELL facts. They correspond to the set:

$$\mathsf{LL}[\mathsf{ELL}] = \{\Pi' \mid \Pi' = [\Pi]^\flat, \text{ for some } \mathsf{2ELL} \text{ proof } \Pi\}$$

7 Why 2-sequents

We discuss in this section some additional features of 2-sequents for linear logic. First, we show a cut-free 2ELL-proof of a formula non cut-free provable in ELL; then, we discuss how *extensions* of Linear Logic may be formulated in this framework.

7.1 An ELL example

System ELL was only sketched in [6]; the complex syntax of the system, however, hides some poison in its tail. Consider indeed the following proof in 2ELL:

$$\dfrac{\dfrac{\dfrac{\vdash A^{\perp\,1}, A^1}{\vdash A^{\perp\,1}, ?A^0}\,? \qquad \dfrac{\vdash 1^1}{\vdash 1^1, ?A^0}\,W}{\vdash A^{\perp} \,\&\, 1^1, ?A^0}\,\&}{\vdash !(A^{\perp} \,\&\, 1)^0, ?A^0}\,!$$

The ending sequent, though derivable in ELL, is not provable in ELL by a cut-free proof, as was observed first by Kanovich, Okada and Scedrov [10]. By inspection of the previous proof, we see that, in order to obtain a cut-free proof of $\vdash !(A^{\perp} \,\&\, 1), ?A$, we should be able to weaken $\vdash 1$ by the formula A, adding the $?$ in front of A only while closing the box of the $!$.

We have not proved yet a complete cut-elimination theorem for the systems presented in this paper (note that in the proof of the correspondence between LL and 2LL we use cut).

7.2 Questing for more freedom

So far, we showed how the 2-sequent approach scales to subsystems of LL obtained constraining the nesting of the corresponding proof-net boxes. In particular, we have seen that these constraints find an immediate translation in 2LL as suitable restrictions of the side-conditions already present in its rules. We also hinted that more drastic restrictions can be made to 2LL, still maintaining completeness for LL (for instance, even restricting all the non-exponential rules to formulas at maximal levels, Corollary 3.9 holds).

On the other hand, rather to strengthen the restrictions already present in 2LL, another possible direction is relaxing or removing the side-conditions. Let us see in details the cases in which this makes sense and which consequences on provability this may have. In particular, we will see that the proposed rules are the most liberal ones for representing LL.

In the rules of 2LL, we have several kinds of constraints: (i) in the promotion rule for $!$; (ii) in the weakening rule; (iii) in the rules for $\forall$ and $\exists$; (iv) in the rules for cut, $\otimes$, and $\oplus$. Let us analyze these cases in order.

7.2.1 Promotion

In 2LL, dropping the constraint of the rule for the ! introduction would kill the exponential structure. The sequent $\vdash ?A \multimap !A$ would become provable (while it is not in

LL). Hence, from $\vdash !A \multimap A$ (valid in LL), we might conclude $A \simeq ?A \simeq !A$.

The previous relaxation has a less drastic effect on the fragments $\mathsf{2LLL}$ and $\mathsf{2ELL}$. For instance, let us take $\mathsf{2ELL}$. The law $\vdash !A \multimap A$ is not derivable in $\mathsf{2ELL}$ and, moreover, we does not obtain it relaxing the constraint on the promotion rule of $\mathsf{2ELL}$. Therefore, in this case the exponential formulas would not collapse to simple ones, but ! and ? would be self-dual, i.e., $!A \simeq ?A$. From a semantical point of view (and from a modal perspective), this would mean that ! is a sort of non-branching next operator; while, from a syntactical point of view, A^i would just become syntactic sugar for $!^iA$ (i.e., A preceded by i occurrences of !).

7.2.2 Weakening

Removing the constraint on weakening would lead to a calculus midway to a linear and affine system (i.e., with unrestricted weakening). In fact, this would correspond to extend the weakening rule of LL to the case of formulas like $!^k?A$, for any $k \geq 0$.

7.2.3 Quantifiers

This seems the least interesting case, since the level constraints are forced by the usual variable constraint on $\forall$. Take, for instace, the following proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash X^{\perp\,1}, X^1}{\vdash ?X^{\perp\,0}, X^1}\,?}{\vdash \exists X.?X^{\perp\,0}, X^1}\,\exists}{\vdash \exists X.?X^{\perp\,0}, \forall X.X^1}\,\forall}{\vdash \exists X.?X^{\perp\,0}, !\forall X.X^0}\,!$$

where the side-condition of $\exists$ is violated. According to the interpretation of levels as boxes, the $\exists$ rule in the proof is outside the box of the !, while the $\forall$ is inside the box. Moreover, the axiom $\vdash X^{\perp\,1}, X^1$ is the only rule inside that box. Therefore, applying $\forall$ we would violate the side-condition on the occurrences of X. In other words, the extension of $\mathsf{2LL}$ corresponding to a free application of $\exists$ is not a logically sound extension of LL.

7.2.4 Multiplicatives and additives

We write $\vdash_{\mathsf{2SLL}}$ for the provability relation obtained allowing an unrestricted use of cut, $\otimes$ and $\oplus$. We have:

$$\vdash_{\mathsf{2SLL}} ?(A^\perp) \oplus ?(B^\perp), !(A \,\&\, B)$$
$$\vdash_{\mathsf{2SLL}} ?(A^\perp) \otimes ?(B^\perp), !(A \parr B)$$

These are essentially the only new sequents introduced by the relaxation. In fact, let S be the the set of axioms (schemas) derived from the pair of sequents above. We have that:

$$\vdash_{\mathsf{2SLL}} A^i \qquad \Longleftrightarrow \qquad \vdash_{\mathsf{LL+S}} A$$

and in particular, when A is exponential free, we have:

$$\vdash_{\mathsf{2SLL}} A^i \qquad \Longleftrightarrow \qquad \vdash_{\mathsf{LL}} A$$

The interesting point is that the axioms in S fits our interpretation of exponentials as a sort of quantifiers. In fact, it is well known that the corresponding sequents in which $\forall$ and $\exists$ replace ! and ? are provable in LL. Namely:

$$\vdash_{\mathsf{LL}} \exists X.(A^{\perp}) \oplus \exists X.(B^{\perp}), \forall X.(A \,\&\, B)$$

$$\vdash_{\mathsf{LL}} \exists X.(A^{\perp}) \otimes \exists X.(B^{\perp}), \forall X.(A \parr B)$$

Unfortunately, the previous extension of LL is not cut-free. From the second sequent in S and from $\vdash_{\mathsf{LL}} ?(A^{\perp} \parr B^{\perp})^0, (A \parr b) \otimes (A \parr B)^0$, we can derive:

$$\frac{\vdash ?A^{\perp} \otimes ?B^{\perp\,0}, !(A \parr B)^0 \qquad \vdash ?(A^{\perp} \parr B^{\perp})^0, (A \parr B) \otimes (A \parr B)^0}{\vdash ?A^{\perp} \parr ?B^{\perp\,0}, (A \parr B) \otimes (A \parr B)^0}\ \mathsf{cut}$$

whose ending sequent is not cut-free derivable.

8 Conclusions and further work

In this paper, our main aim has been showing how the 2-sequent approach may improve the presentation of calculi whose rules encode involved structural constraints on the corresponding proofs. Because of this, we omitted to study the dynamics of the systems that we proposed and we focused on examples and observations made possible by the use of our generalized notion of sequent.

The most relevant point of this approach to Linear Logic is the tight correspondence between the indexes assigned to formulas and the box nesting depth of links in the corresponding proof-nets. Therefore, the further step is the analysis of the indexed proof-nets induced by our systems and, in particular, a detailed study of their reduction. As we showed in [7, 9], the dynamics of the indexed proof-nets corresponding to 2-sequents of LL can be implemented via a set of local an distributed rules (i.e., no more global rules for duplication of boxes). We believe that this approach might scale in a smooth way to the systems presented here. Moreover, we hope that this reduction technique might preserve the complexity bound that motivated Girard, and we think that sharing-reduction will give a clearer explanation of these bounds.

A Light and Elementary Linear Logic

The rules of LLL are given in Figure 3. Remind that $\mathbf{A}$ stands for a block of formulas $A_1, \ldots, A_n$. Moreover, $A_1, \ldots, A_n$ "is hypocrisy for" $A_1 \oplus \ldots \oplus A_n$, while if $\mathbf{A}_1, \ldots, \mathbf{A}_n$ stand for the formulas $A_1, \ldots, A_n$, the sequence $\mathbf{A}_1; \ldots; \mathbf{A}_n$ "is hypocrisy for" for $A_1 \parr \ldots \parr A_n$.

For a complete treatment, including the relevant issues of the dynamics of LLL, see [6] (for a semantical approach, see [10]).

ELL is obtained from LLL replacing the following of-course

$$\frac{\vdash C_1^1, \ldots, C_{p_1}^1; \ldots; C_1^n, \ldots, C_{p_n}^n; A}{\vdash [C_1^1]; \ldots; [C_{p_1}^1]; \ldots; [C_1^n]; \ldots; [C_{p_n}^n]; !A}\ \text{of-course}$$

for the one of LLL. At the same time the modality § becomes superfluous and is removed from the calculus.

Identity/Negation

$$\frac{}{\vdash A;\ A^{\perp}}\ \mathsf{axiom} \qquad \frac{\vdash \Gamma;\ A \quad \vdash \Delta;\ A^{\perp}}{\vdash \Gamma;\ \Delta}\ \mathsf{cut}$$

Structure

$$\frac{\vdash \Gamma}{\vdash \Gamma; [A]}\ \mathsf{mult\text{-}W} \quad \frac{\vdash \Gamma;\ \mathbf{A}}{\vdash \Gamma;\ \mathbf{A}, B}\ \mathsf{add\text{-}W} \qquad \frac{\vdash \Gamma;\ [A];\ [A]}{\vdash \Gamma;\ [A]}\ \mathsf{mult\text{-}C} \quad \frac{\vdash \Gamma;\ \mathbf{A}, B, B}{\vdash \Gamma;\ \mathbf{A}, B}\ \mathsf{add\text{-}C}$$

Logic

$$\frac{\vdash B_1, \ldots, B_n;\ A}{\vdash [B_1]; \ldots; [B_n];\ !A}\ \mathsf{of\text{-}course}_{\,n>0} \qquad \frac{\vdash \Gamma;\ A}{\vdash \Gamma;\ ?A}\ \mathsf{why\text{-}not}$$

$$\frac{\vdash C_1^1, \ldots, C_{p_1}^1; \ldots; C_1^n, \ldots, C_{p_n}^n; A_1; \ldots; A_m}{\vdash [C_1^1]; \ldots; [C_{p_1}^1]; \ldots; [C_1^n]; \ldots; [C_{p_n}^n];\ \S A_1; \ldots;\ \S A_m}\ \mathsf{neutral}$$

$$\frac{\vdash \Gamma;\ C \quad \vdash \Gamma;\ D}{\vdash \Gamma;\ C \,\&\, D}\ \mathsf{with} \qquad \frac{\vdash \Gamma;\ C}{\vdash \Gamma;\ C \oplus D}\ \mathsf{L\text{-}plus} \quad \frac{\vdash \Gamma;\ D}{\vdash \Gamma;\ C \oplus D}\ \mathsf{R\text{-}plus}$$

$$\frac{\vdash \Gamma;\ C \quad \vdash \Delta;\ D}{\vdash \Gamma;\ \Delta;\ C \otimes D}\ \mathsf{tensor} \qquad \frac{\vdash \Gamma;\ C;\ D}{\vdash \Gamma;\ C \parr D}\ \mathsf{par}$$

$$\frac{\vdash \Gamma;\ A}{\vdash \Gamma;\ \forall X.A}\ \mathsf{forall} \qquad \frac{\vdash \Gamma;\ A[B/X]}{\vdash \Gamma;\ \exists X.A}\ \mathsf{exists}$$

$$\frac{\vdash \Gamma}{\vdash \Gamma;\ \perp}\ \mathsf{false} \qquad \frac{}{\vdash 1}\ \mathsf{one} \qquad \frac{}{\vdash \Gamma;\ \top}\ \mathsf{true}$$

FIG. 3. Girard's LLL

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