Modal Logic, Linear Logic, Optimal Lambda-reduction

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0. Thesis

A (purist) approach to modal proof-theory can have some (applied) benefits for the efficient reduction of lambda-terms, a problem of paramount importance in the implementation of functional programming languages.

1. Modal Proof-theory

The shape of the rules of standard proof-theoretical systems for modal logics is often an example of what a good rule should not be. The intensionality of the modal connectives forces the rules either to be non symmetric (*e.g.* the sequent rule for necessity in $\mathsf{S4}$) or to involve a global constraint on the shape of the deduction (*e.g.* Prawitz's "third" formulation of the natural deduction introduction rule for necessity in $\mathsf{S4}$). To restore symmetry and locality, several proposals have been made to extend the notion of sequent, among which the *2-sequents* of [9]. This proposal, originally formulated as a sequent calculus for the (classical) logic KD, has then been extended to natural deduction for intuitionistic KD [10], intuitionistic $\mathsf{S4}$ [7], and linear logic [8]. In this context a sequent has the form $A_1^{k_1}, \ldots, A_n^{k_n} \vdash B^k$, where the k's are positive indexes (the *levels*), with $k_i \leq k$, for any $1 \leq i \leq n$. If Γ^{k_i} denotes all the assumptions at level k_i, the intended interpretation of the above sequent is $\bigwedge \Gamma^1 \supset \Box(\bigwedge \Gamma^2 \supset \ldots \Box(\bigwedge \Gamma^k \supset B) \ldots)$. Levels express in the calculus a form of modal dependency. The problematic rules (necessity introduction, for instance) can now be expressed with constraints on (and manipulation of) levels, in a local and symmetric way (see [8] for a full presentation of the system, proofs of its main properties, and the relations with Prawitz's approach).

2. Optimal Lambda-reduction

There is no recursive optimal one-step (*i.e.* selecting a single redex at the time) strategy for beta-reduction. To obtain optimal strategies, [6] introduces family reductions, reducing in a single step suitable sets of redexes and avoiding, in this, any unnecessary duplication. The implementation of such an optimal strategy has been an open problem until [5] and [3], gave a graph-theoretic rewriting system. The maintenance of the families, however, requires additional information to be added to the representation of the lambda-terms, in the form of *control nodes*. During reduction, some of these nodes become unnecessary and should be erased. To this end, however, the rewriting rules are not sufficient: There are graphs in normal form with control nodes that, semantically, are unnecessary. This proliferation of nodes, besides being theoretically unpleasant, potentially slows down the reduction process.

3. Levels in Optimal Reduction

When the 2-sequent approach is tailored to linear logic and it is seen in a proof-net framework, the levels appear as a very natural information (they correspond to the number of boxes a formula is contained into). The insight we obtained from [8] —building on the relations between optimal reduction implementations and linear logic discovered in [4] and developed in [1, 2]— gives us the opportunity to improve the graph reduction implementation, obtaining a system where:

(i) A graph in normal form with respect to beta-reduction has no spurious control nodes.

(ii) A graph in normal form with respect to all rules is the representation of the normal form of the original lambda-term. The rewriting system is confluent and normalising.

(iii) The erasing of subterms can be easily treated (a subject which has been neglected in all the relevant literature).

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